

Inverse source problem for time fractional diffusion with discrete random noise

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Abstract

In this paper, we deal with the inverse source problem of determining a source in a time fractional diffusion equation where data are given at a fixed time. This problem is ill-posed, i.e., the solution does not depend continuously on the data. To regularize the instable solution, we use the trigonometric method in nonparametric regression associated with the truncated expansion method. We also investigate the convergence rate.

1 Introduction

In this work, we focus on an inverse problem for the following time-fractional diffusion equation with a source:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F(x, t), & (x, t) \in (0, \pi) \times (0, T) \\ u_x(0, t) = u_x(\pi, t) = 0, \\ u(x, 0) = 0, & x \in \Omega \end{cases} \quad (1.1)$$

where $\Omega = (0, \pi)$, $T > 0$ and $0 < \alpha < 1$. Here $\frac{\partial^\alpha u}{\partial t}$ is the Caputo fractional derivative of order α derivative which first appeared in [2] and is defined for an absolutely continuous function u as

$$\frac{\partial^\alpha u(t)}{\partial t} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds, \quad (1.2)$$

where Γ denotes the standard Gamma function. Note that when the fractional order α is equal to 1, the fractional derivative $\frac{\partial^\alpha u}{\partial t}$ is equal to the first-order derivative $\frac{du}{dt}$ [7], and thus the problem (1.1) becomes the classical diffusion problem. When $F(x, t) \equiv 0$, the problem (1.1) was first studied by Nigmatullin [12], and [20]. Recently, among many other researchers, Meerchaert et al. [10] and Baeumer et al. [1] have studied the problem (1.1) in a bounded domain in \mathbb{R}^d . They also obtained a probabilistic representation of the solution using a time-changed Brownian motion, or other time-changed processes.

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Problem (1.1) is a forward problem when the source function $F = F(x, t)$ is given appropriately. Whereas, an inverse source problem based on problem (1.1) is to determine the source term F at a previous time from its value at a final time T as follows:

$$u(x, T) = u_T(x), \quad x \in \Omega.$$

where the source function $F = F(x, t)$ can be split into a product $R(t)f(x)$, and $R(t)$ is known in advance.

It is known that the inverse source problem mentioned above is ill-posed in general, i.e., a solution does not always exist. When the solution exists, the solution does not depend continuously on the given initial data. In fact, from a small noise of a physical measurement, the corresponding solution may have a large error. This makes the numerical computation troublesome. Hence a regularization is required.

If $\alpha = 1$, the inverse source problem (1.1) is the classical ill-posed problem and has been studied in [5, 6]. However, there are only a few works on the fractional inverse source problem; for example, Sakamoto et al. [15] used the data $u(x_0, t)$ ($x_0 \in \Omega$) to determine $R(t)$ once $f(x)$ was given, where the authors obtained a Lipschitz stability for $R(t)$. Wei et al. [19] used the Fourier truncation method to solve an inverse source problem with $R(t) = 1$ in the problem (1.1) for one-dimensional problem with special coefficients. Actually, there are very limited number of results for the inverse source problem for the time-fractional diffusion equation when $R(t)$ depends on time.

Murio [11] considered an inverse problem of recovering boundary functions from transient data at an interior point in a 1-D semi-infinite half-order time-fractional diffusion equation. Liu and Yamamoto [9] applied a quasi-reversibility regularization method to solve a backward problem for the time-fractional diffusion equation. Kirane and Malik [8] considered an inverse source problem but they didn't study regularization problem. Recently, among many others, Jin and Rundell [4], Wei and Wang [17] and Wnag et al. [18] have studied an inverse problem for the time fractional diffusion. Tuan et al [16] considered the inverse source problem (1.1) for the deterministic case using Tikhonov regularization.

As is well-known, measurements always are given at a discrete set of points and contain errors. These errors may be generated from controllable sources or uncontrollable sources. In the first case, the error is often deterministic. Hence, if we know an approximation u_T^ϵ of the final data u_T then we can construct an approximation f^ϵ for the function f . If the errors are generated from uncontrollable sources as wind, rain, humidity, etc, then the model is random. Methods used for the deterministic cases cannot be applied directly to the random case. Because of the random noise, the calculation is often intractable. In practical situations, the function $u_T(x)$ is a result of experimental measurements and it cannot be observed without errors: hence, in general we have

$$\tilde{u}_T(x_k) = u_T(x_k) + \epsilon_k, \quad k = 1, \dots, n \quad (1.3)$$

where $\epsilon_k, k = 1, \dots, n$ are unknown independent random errors. In fact, these errors can come from many sources like the measuring instrument or the environment.

From now on, we put

$$x_k = \pi \frac{2k-1}{2n}, \quad \text{with } k = 1, \dots, n.$$

We have a data set $D = (\tilde{u}_T(x_1), \tilde{u}_T(x_2), \dots, \tilde{u}_T(x_n))$, which is the measure of

$$(u_T(x_1), u_T(x_2), \dots, u_T(x_n)).$$

Here D satisfies

$$\widetilde{u}_T(x_k) = u_T(x_k) + \sigma_k \epsilon_k, \quad (1.4)$$

where, $\epsilon_k, k = 1, \dots, n$ are unknown independent noises. Hence $\epsilon_k \sim \mathcal{N}(0, 1)$, and σ_k are unknown positive constants which are bounded by a positive constant V_{\max} , i.e., $0 \leq \sigma_k < V_{\max}$ for all $k = 1, \dots, n$. The noises ϵ_k are mutually independent. To the best of our knowledge, there does not exist any results on inverse source problem for fractional diffusion with random noise in the literature. Our main problem in this paper is finding the source function f from the random data $u_T(x_k), k = 1, \dots, n$.

Our main result in this paper is the following theorem.

Theorem 1.1. *Let $\epsilon > 0$ and $\epsilon_k \sim N(0, 1)$ be independent normal random variables with $k = 1, \dots, n$ (as mentioned above). Assume that there exists $\beta > 0$ and $E > 0$ such that*

$$\|f\|_{H^\beta(\Omega)} \leq E, \quad (1.5)$$

where $\|f\|_{H^\beta(\Omega)}^2 = \sum_{p=1}^{\infty} p^{2\beta} |\langle f, \phi_p \rangle|^2$. Then a regularized function $\widetilde{f}_{n,M}$ for f can be computed as follows

$$\widetilde{f}_{n,M}(x) = \frac{\frac{1}{n} \sum_{k=1}^n \widetilde{u}_T(x_k)}{\int_0^T (T-s)^{\alpha-1} R(s) ds} + \sum_{p=1}^M \frac{\frac{\pi}{n} \sum_{k=1}^n \widetilde{u}_T(x_k) \phi_p(x_k)}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds} \phi_p(x)$$

where $\phi_p(x) = \sqrt{\frac{2}{\pi}} \cos(px)$ for $p = 0, 1, 2, 3, \dots$ is a sequence of an orthonormal basis of $L^2(\Omega)$ and the natural numbers n, M are called regularization parameters. Moreover, we have the following estimate

$$\begin{aligned} \mathbb{E} \left\| \widetilde{f}_{n,M}(x) - f(x) \right\|_{L^2(\Omega)}^2 &\leq \frac{(2-\alpha)^2}{R_0^2 T^{4-2\alpha}} \left(\frac{\pi^2 V_{\max}^2}{n} + \frac{\pi^3 \|R\|_\infty^2 E^2}{288 n^4} \right) \\ &\quad + \frac{1}{R_0^2 [1 - E_{\alpha,1}(-T^\alpha)]^2} \left(\frac{\pi^2 V_{\max}^2}{n} + \frac{\pi^4 \|R\|_\infty^2 E^2}{144 n^4} \right) M^5 \\ &\quad + M^{-2\beta} E^2. \end{aligned} \quad (1.6)$$

Let $M := M_n$ such that $0 < M := M_n < n$ and

$$\lim_{n \rightarrow +\infty} \frac{M^5}{n} = 0 \quad (1.7)$$

then

$$\mathbb{E} \left\| \widetilde{f}_{n,M}(x) - f(x) \right\|_{L^2(\Omega)}^2 \text{ is of order } \max \left(\frac{M^5}{n}, M^{-2\beta} \right). \quad (1.8)$$

Remark 1.1. By choosing $M := n^{\frac{1}{5+2\beta}}$, and by (1.8), we can conclude that

$$\mathbb{E} \left\| \widetilde{f}_{n,M}(x) - f(x) \right\|_{L^2(\Omega)}^2 \text{ is of order } \left(\frac{1}{n} \right)^{\frac{2\beta}{5+2\beta}}. \quad (1.9)$$

Remark 1.2. *The methods of proof in this paper can be used to handle the equation with Laplacian in higher dimensions and Lévy operators corresponding to a wide range of Lévy processes. In particular, this corresponds to replacing the second derivative in equation (1.1) with Laplacian, or a fractional Laplacian in a bounded domain in \mathbb{R}^d with Dirichlet or Neuman boundary conditions.*

We give the proof of the main result in the next section. We prove a sequence of Lemmas that are of interest in their own right as well.

2 Proof of main results

The proof of the main result follows from a couple of Lemmas.

Lemma 2.1. *(see [13]) Let $\lambda > 0$, then we have:*

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0, 0 < \alpha < 1.$$

Lemma 2.2. *Let $R : [0, T] \rightarrow \mathbb{R}$ be a positive continuous function such that $\inf_{t \in [0, T]} |R(t)| = R_0 > 0$. Set $\|R\|_\infty = \sup_{t \in [0, T]} |R(t)|$. Then we have for all $p \in \mathbb{N}$*

$$\frac{R_0[1 - E_{\alpha,1}(-T^\alpha)]}{p^2} \leq \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds \leq \frac{\|R\|_\infty}{p^2}.$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned} \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) ds &= \int_0^T \left| s^{\alpha-1} E_{\alpha,\alpha}(-p^2 s^\alpha) \right| ds = -\frac{1}{p^2} \int_0^T \frac{d}{ds} E_{\alpha,1}(-p^2 s^\alpha) ds \\ &= \frac{1 - E_{\alpha,1}(-p^2 T^\alpha)}{p^2} \leq \frac{1}{p^2}. \end{aligned}$$

This implies that

$$\begin{aligned} \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds &\leq \sup_{t \in [0, T]} |R(t)| \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) ds \\ &\leq \frac{\|R\|_\infty}{p^2}. \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds &\geq \left(\inf_{t \in [0, T]} |R(t)| \right) \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) ds \\ &\geq R_0 \frac{1 - E_{\alpha,1}(-p^2 T^\alpha)}{p^2} \geq R_0 \frac{1 - E_{\alpha,1}(-T^\alpha)}{p^2}. \end{aligned} \tag{2.11}$$

in the last inequality we use the fact that $E_{\alpha,1}(-p^2 T^\alpha) \leq E_{\alpha,1}(-T^\alpha)$ for $p \geq 1$; see, for example, [14]. The proof is completed. \square

Lemma 2.3 ([3], page 144). Let $p = 1, \dots, n-1$, and $q = 1, 2, \dots$, with $x_k = \pi \frac{2k-1}{2n}$ and

$\phi_p(x_k) = \sqrt{\frac{2}{\pi}} \cos(px_k)$, then we have

$$s_{p,q} = \frac{\sum_{k=1}^n \phi_p(x_k) \phi_q(x_k)}{n} = \begin{cases} \frac{1}{\pi}, & q-p = 2ln \text{ or } q+p = 2ln \text{ (} l \text{ even)}, \\ -\frac{1}{\pi}, & q-p = 2ln \text{ or } q+p = 2ln \text{ (} l \text{ odd)}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.12)$$

If $q = 1, 2, \dots, n-1$, then

$$s_{p,q} = \begin{cases} \frac{1}{\pi}, & p = q, \\ 0, & p \neq q. \end{cases} \quad (2.13)$$

and

$$\frac{1}{n} \sum_{k=1}^n \phi_p(x_k) = \begin{cases} 0, & p \neq 2ln, \\ (-1)^l \sqrt{\frac{2}{\pi}}, & p = 2ln. \end{cases}$$

From this lemma, we have the next result.

Lemma 2.4. Let $p, n \in \mathbb{N}$ such that $0 \leq p \leq n-1$. Assume that u_T is piecewise C^1 on $[0, \pi]$. Then

$$\langle u_T(x), \phi_p(x) \rangle = \begin{cases} \frac{1}{n} \sum_{k=1}^n u_T(x_k) - \tilde{G}_{n0}, & p = 0, \\ \frac{\pi}{n} \sum_{k=1}^n u_T(x_k) \phi_p(x_k) - \tilde{G}_{np}, & 1 \leq p \leq n-1. \end{cases} \quad (2.14)$$

where

$$\tilde{G}_{np} = \begin{cases} \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} (-1)^l \langle u_T(x), \phi_{2ln}(x) \rangle, & p = 0, \\ \sum_{l=1}^{\infty} (-1)^l \left[\langle u_T(x), \phi_{p+2ln}(x) \rangle + \langle u_T(x), \phi_{-p+2ln}(x) \rangle \right], & 1 \leq p \leq n-1. \end{cases} \quad (2.15)$$

Proof. Using the complete orthonormal basis $\{\phi_p; p = 0, 1, 2, \dots\}$, the function u_T can be written as follows

$$u_T(x_k) = \langle u_T(x), \phi_0(x) \rangle + \sum_{q=1}^{\infty} \langle u_T(x), \phi_q(x) \rangle \phi_q(x_k).$$

This implies that

$$\frac{1}{n} \sum_{k=1}^n u_T(x_k) \phi_p(x_k) = \frac{1}{n} \sum_{k=1}^n \langle u_T(x), \phi_0(x) \rangle \phi_p(x_k) + \frac{1}{n} \sum_{k=1}^n \left(\sum_{q=1}^{\infty} \langle u_T(x), \phi_q(x) \rangle \phi_q(x_k) \right) \phi_p(x_k).$$

For $p \geq 1$, using Lemma 2.3, we get

$$\begin{aligned}
\frac{1}{n} \sum_{k=1}^n u_T(x_k) \phi_p(x_k) &= \frac{1}{n} \sum_{k=1}^n \left(\sum_{q=1}^n \langle u_T(x), \phi_q(x) \rangle \phi_q(x_k) \right) \phi_p(x_k) \\
&\quad + \frac{1}{n} \sum_{k=1}^n \left(\sum_{q=n+1}^{\infty} \langle u_T(x), \phi_q(x) \rangle \phi_q(x_k) \right) \phi_p(x_k) \\
&= \frac{1}{n} \sum_{q=1}^n \langle u_T(x), \phi_q(x) \rangle \left[\sum_{k=1}^n \phi_q(x_k) \phi_p(x_k) \right] \\
&\quad + \frac{1}{n} \sum_{q=n+1}^{\infty} \langle u_T(x), \phi_q(x) \rangle \left[\sum_{k=1}^n \phi_q(x_k) \phi_p(x_k) \right]. \tag{2.16}
\end{aligned}$$

By Lemma 2.3, we obtain for $1 \leq q \leq n$

$$\langle u_T(x), \phi_q(x) \rangle \left[\sum_{k=1}^n \phi_q(x_k) \phi_p(x_k) \right] = \begin{cases} \frac{\langle u_T(x), \phi_q(x) \rangle n}{\pi}, & p = q, \\ 0, & p \neq q. \end{cases} \tag{2.17}$$

For $q \geq n$

$$\langle u_T(x), \phi_q(x) \rangle \left[\sum_{k=1}^n \phi_q(x_k) \phi_p(x_k) \right] = \begin{cases} \frac{(-1)^l n \langle u_T(x), \phi_q(x) \rangle}{\pi}, & \text{if } q = p + 2ln \text{ or } q = -p + 2ln, \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

Hence

$$\frac{1}{n} \sum_{q=1}^n \langle u_T(x), \phi_q(x) \rangle \left[\sum_{k=1}^n \phi_q(x_k) \phi_p(x_k) \right] = \frac{\langle f(x), \phi_p(x) \rangle}{\pi}. \tag{2.19}$$

This implies that

$$\frac{1}{n} \sum_{q=n+1}^{\infty} \langle u_T(x), \phi_q(x) \rangle \left[\sum_{k=1}^n \phi_q(x_k) \phi_p(x_k) \right] = \frac{1}{\pi} \sum_{l=1}^{\infty} (-1)^l \left[\langle u_T(x), \phi_{p+2ln}(x) \rangle + \langle u_T(x), \phi_{-p+2ln}(x) \rangle \right]. \tag{2.20}$$

Combining (2.18), (2.19), (2.20) we get

$$\frac{1}{n} \sum_{k=1}^n u_T(x_k) \phi_p(x_k) = \frac{1}{\pi} \left[\langle u_T(x), \phi_p(x) \rangle + \tilde{G}_{np} \right]. \tag{2.21}$$

Therefore, (2.15) holds for $1 \leq p \leq n-1$. Similarly, we have

$$\begin{aligned}
\sum_{k=1}^n u_T(x_k) &= n \langle u_T(x), \phi_0(x) \rangle + \sum_{p=1}^{\infty} \langle u_T(x), \phi_p(x) \rangle \sum_{k=1}^n \phi_p(x_k) \\
&= n \langle u_T(x), \phi_0(x) \rangle + n \tilde{G}_{n0}.
\end{aligned}$$

This completes the proof of Lemma 2.4. \square

The following Lemma gives the formula of f in terms of $u_T(x_k)$.

Lemma 2.5. *Let $0 < M < n$, $M \in \mathbb{N}$. Assume that u_T is as in Lemma 2.4. Then the source function f is given by*

$$\begin{aligned} f(x) = & \frac{\frac{1}{n} \sum_{k=1}^n u_T(x_k) - \tilde{G}_{n0}}{\int_0^T (T-s)^{\alpha-1} R(s) ds} + \sum_{p=1}^M \frac{\frac{\pi}{n} \sum_{k=1}^n u_T(x_k) \phi_p(x_k) - \tilde{G}_{np}}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds} \phi_p(x) \\ & + \sum_{p=M+1}^{\infty} \frac{\langle u_T(x), \phi_p(x) \rangle}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds} \phi_p(x). \end{aligned} \quad (2.22)$$

Proof. First, we have the following equality

$$f(x) = \frac{\langle u_T(x), \phi_0(x) \rangle}{\int_0^T (T-s)^{\alpha-1} R(s) ds} + \sum_{p=1}^{\infty} \frac{\langle u_T(x), \phi_p(x) \rangle \phi_p(x)}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds}. \quad (2.23)$$

To prove (2.23), we use the results of Sakamoto and Yamamoto [15]. According Theorem 2.4 in [15], the solution of (1.1) satisfies

$$\langle u(x, t), \phi_p(x) \rangle = E_{\alpha,1}(-p^2 t^\alpha) \langle u(x, 0), \phi_p(x) \rangle + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(t-s)^\alpha) \langle F(x, s), \phi_p(x) \rangle ds. \quad (2.24)$$

By letting $t = T$ in the last equality, recalling $u(x, 0) = 0$ and $F_p(s) = R(s) \langle f(x), \phi_p(x) \rangle$, we get

$$\langle u(x, T), \phi_p(x) \rangle = \langle f(x), \phi_p(x) \rangle \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds. \quad (2.25)$$

This implies (2.23).

Using Lemma 2.4, the first term of the right hand side of the last equality is equal to

$$\langle f(x), \phi_0(x) \rangle = \frac{\langle u_T(x), \phi_0(x) \rangle}{\int_0^T (T-s)^{\alpha-1} R(s) ds} = \frac{\frac{1}{n} \sum_{k=1}^n u_T(x_k) - \tilde{G}_{n0}}{\int_0^T (T-s)^{\alpha-1} R(s) ds} \quad (2.26)$$

and the second term on the right hand side of (2.23) is equal to

$$\begin{aligned}
& \sum_{p=1}^{\infty} \frac{\langle u_T(x), \phi_p(x) \rangle \phi_p(x)}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} \\
&= \sum_{p=1}^M \frac{\langle u_T(x), \phi_p(x) \rangle \phi_p(x)}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} + \sum_{p=M+1}^{\infty} \frac{\langle u_T(x), \phi_p(x) \rangle \phi_p(x)}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} \\
&= \sum_{p=1}^M \frac{\left(\frac{\pi}{n} \sum_{k=1}^n u_T(x_k) \phi_p(x) - \tilde{G}_{np} \right) \phi_p(x)}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} + \sum_{p=M+1}^{\infty} \frac{\langle u_T(x), \phi_p(x) \rangle \phi_p(x)}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds}.
\end{aligned} \tag{2.27}$$

Substituting (2.27) into (2.23), we get (2.22). \square

Now, we return the proof of main result.

Proof of Theorem 1.1. First, we have the following estimate

$$\begin{aligned}
\left| \langle u_T(x), \phi_p(x) \rangle \right| &= \left| \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds \right| \left| \langle f(x), \phi_p(x) \rangle \right| \\
&\leq \frac{\|R\|_{\infty} \|f\|_{L^2(\Omega)}}{p^2}.
\end{aligned} \tag{2.28}$$

Using (1.6) and (2.23), we obtain

$$\begin{aligned}
\tilde{f}_{n,M}(x) - f(x) &= \frac{\frac{1}{n} \sum_{k=1}^n \sigma_k \epsilon_k + \tilde{G}_{n0}}{\int_0^T (T-s)^{\alpha-1} R(s) ds} + \sum_{p=1}^M \left[\frac{\frac{\pi}{n} \sum_{k=1}^n \sigma_k \epsilon_k \phi_p(x_k) + \tilde{G}_{np}}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} \right] \phi_p(x) \\
&\quad - \sum_{p=M+1}^{\infty} \left[\frac{\langle u_T(x), \phi_p(x) \rangle}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} \right] \phi_p(x)
\end{aligned} \tag{2.29}$$

Applying Lemma 2.5 we obtain

$$\begin{aligned}
\|\tilde{f}_{n,M}(x) - f(x)\|_{L^2(\Omega)}^2 &= \left[\frac{\frac{1}{n} \sum_{k=1}^n \sigma_k \epsilon_k + \tilde{G}_{n0}}{\int_0^T (T-s)^{\alpha-1} R(s) ds} \right]^2 + \sum_{p=1}^M \left[\frac{\frac{\pi}{n} \sum_{k=1}^n \sigma_k \epsilon_k \phi_p(x_k) + \tilde{G}_{np}}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} \right]^2 \\
&\quad + \sum_{p=M+1}^{\infty} \left[\frac{\langle u_T(x), \phi_p(x) \rangle}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^{\alpha}) R(s) ds} \right]^2
\end{aligned}$$

This follows from the Parseval identity and the fact that $\mathbb{E}(\epsilon_j \epsilon_l) = 0$; ($j \neq l$), and $\mathbb{E} \epsilon_j = 0$; $j = 1, 2, \dots, n$. Now

$$\begin{aligned}
& \mathbb{E} \left\| \tilde{f}_{n,M}(x) - f(x) \right\|_{L^2(\Omega)}^2 \\
&= \underbrace{\frac{1}{n^2} \sum_{k=1}^n \sigma_k^2 \mathbb{E} \epsilon_k^2 + \tilde{G}_{n0}^2}_{=: I_1} + \underbrace{\sum_{p=M+1}^{\infty} \left[\frac{\langle u_T(x), \phi_p(x) \rangle}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds} \right]^2}_{=: I_2} \\
&+ \underbrace{\sum_{p=1}^M \frac{\frac{\pi^2}{n^2} \sum_{k=1}^n \sigma_k^2 \mathbb{E} \epsilon_k^2 + \tilde{G}_{np}^2}{\left[\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds \right]^2}}_{=: I_3}. \tag{2.30}
\end{aligned}$$

First, by (2.15) and (2.28) we know that

$$\tilde{G}_{n0} \leq \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} |\langle u_T(x), \phi_{2ln}(x) \rangle| \leq \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} \frac{\|R\|_{\infty} \|f\|_{L^2(\Omega)}}{4l^2 n^2} = \sqrt{\frac{2}{\pi}} \frac{\pi^2}{24} \frac{\|R\|_{\infty} \|f\|_{L^2(\Omega)}}{n^2}. \tag{2.31}$$

where we use the fact that $\sum_{l=1}^{\infty} \frac{1}{l^2} = \frac{\pi^2}{6}$. By a similar method as above, we obtain

$$\begin{aligned}
\tilde{G}_{np} &\leq \sum_{l=1}^{\infty} \left| \langle u_T(x), \phi_{p+2ln}(x) \rangle + \langle u_T(x), \phi_{-p+2ln}(x) \rangle \right| \\
&\leq \|R\|_{\infty} \|f\|_{L^2(\Omega)} \left[\sum_{l=1}^{\infty} \frac{1}{(p+2ln)^2} + \sum_{l=1}^{\infty} \frac{1}{(-p+2ln)^2} \right] \\
&\leq \frac{\pi^2}{12} \frac{\|R\|_{\infty} \|f\|_{L^2(\Omega)}}{n^2}. \tag{2.32}
\end{aligned}$$

Since $\sigma_k < V_{\max}$, we estimate I_1 as follows

$$I_1 \leq \frac{\frac{V_{\max}^2}{n} + \tilde{G}_{n0}^2}{\pi^2 \left(\int_0^T (T-s)^{\alpha-1} R(s) ds \right)^2} \leq \frac{(2-\alpha)^2}{R_0^2 T^{4-2\alpha}} \left(\frac{V_{\max}^2}{n} + \frac{\pi^3}{288} \frac{\|R\|_{\infty}^2 \|f\|_{L^2(\Omega)}^2}{n^4} \right). \tag{2.33}$$

By equation (2.23), we know that for $p \geq 1$

$$\langle f(x), \phi_p(x) \rangle = \frac{\langle u_T(x), \phi_p(x) \rangle}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds}. \tag{2.34}$$

Recall the definition of I_2 in (2.30):

$$I_2 = \sum_{p=M+1}^{\infty} \left[\frac{\langle u_T(x), \phi_p(x) \rangle}{\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds} \right]^2. \tag{2.35}$$

using the last two equations, we get

$$I_2 = \sum_{p=M+1}^{\infty} \left[\langle f(x), \phi_p(x) \rangle \right]^2. \quad (2.36)$$

Since $1 = p^{-2\beta} p^{2\beta}$, we can rewrite I_2 as follows

$$I_2 = \sum_{p=M+1}^{\infty} p^{-2\beta} p^{2\beta} \left| \langle f(x), \phi_p(x) \rangle \right|^2. \quad (2.37)$$

In the last series (2.37) since $p \geq M+1 > M$, we get $p^{-2\beta} \leq M^{-2\beta}$.

Using the last two observations, we obtain

$$I_2 \leq \sum_{p=M+1}^{\infty} M^{-2\beta} p^{2\beta} \left| \langle f(x), \phi_p(x) \rangle \right|^2 \phi_p^2(x) = M^{-2\beta} \underbrace{\sum_{p=M+1}^{\infty} p^{2\beta} \left| \langle f(x), \phi_p(x) \rangle \right|^2}_{:= \tilde{I}_2}. \quad (2.38)$$

It is easy to see that

$$\tilde{I}_2 = \sum_{p=M+1}^{\infty} p^{2\beta} \left| \langle f(x), \phi_p(x) \rangle \right|^2 \leq \sum_{p=1}^{\infty} p^{2\beta} \left| \langle f(x), \phi_p(x) \rangle \right|^2 = \|f\|_{H^\beta(\Omega)}^2. \quad (2.39)$$

Using (2.38) and (2.39), we get

$$I_2 \leq M^{-2\beta} \|f\|_{H^\beta(\Omega)}^2. \quad (2.40)$$

Using Lemma 2.2, the I_3 term can be estimated as follows

$$\begin{aligned} I_3 &\leq \left(\frac{\pi^2 V_{\max}^2}{n} + \frac{\pi^4}{144} \frac{\|R\|_{\infty}^2 \|f\|_{L^2(\Omega)}^2}{n^4} \right) \sum_{p=1}^M \left[\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-p^2(T-s)^\alpha) R(s) ds \right]^{-2} \\ &\leq \left(\frac{\pi^2 V_{\max}^2}{n} + \frac{\pi^4}{144} \frac{\|R\|_{\infty}^2 \|f\|_{L^2(\Omega)}^2}{n^4} \right) \sum_{p=1}^M \frac{p^4}{R_0^2 [1 - E_{\alpha,1}(-T^\alpha)]^2} \\ &\leq \frac{1}{R_0^2 [1 - E_{\alpha,1}(-T^\alpha)]^2} \left(\frac{\pi^2 V_{\max}^2}{n} + \frac{\pi^4 \|R\|_{\infty}^2 \|f\|_{L^2(\Omega)}^2}{144 n^4} \right) M^5. \end{aligned} \quad (2.41)$$

Combining (2.30), (2.33), (2.40), (2.41), we obtain

$$\begin{aligned} \mathbb{E} \left\| \tilde{f}_{n,M}(x) - f(x) \right\|_{L^2(\Omega)}^2 &\leq \frac{(2-\alpha)^2}{R_0^2 T^{4-2\alpha}} \left(\frac{\pi^2 V_{\max}^2}{n} + \frac{\pi^3}{288} \frac{\|R\|_{\infty}^2 \|f\|_{L^2(\Omega)}^2}{n^4} \right) \\ &\quad + \frac{1}{R_0^2 [1 - E_{\alpha,1}(-T^\alpha)]^2} \left(\frac{\pi^2 V_{\max}^2}{n} + \frac{\pi^4 \|R\|_{\infty}^2 \|f\|_{L^2(\Omega)}^2}{144 n^4} \right) M^5 \\ &\quad + M^{-2\beta} \|f\|_{H^\beta}^2. \end{aligned} \quad (2.42)$$

This completes the proof. \square

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References

- [1] B. Baeumer, T. Luks, and M.M. Meerschaert. Space-time fractional Dirichlet problems. Preprint available at <http://www.stt.msu.edu/users/mcubed/SpaceTimeFrac.pdf>
- [2] M. Caputo. Linear models of dissipation whose Q is almost frequency independent, Part II. *Geophys. J. R. Astr. Soc.* 13 (1967), 529-539.
- [3] R.L. Eubank, *Nonparametric regression and spline smoothing*, Second edition. Statistics: Textbooks and Monographs, 157. Marcel Dekker, Inc., New York, 1999. xii+338 pp. ISBN: 0-8247-9337-4
- [4] B. Jin, W. Rundell, *An inverse problem for a one-dimensional time-fractional diffusion problem* Inverse Problems 28 (2012), no. 7, 075010, 19 pp
- [5] A. Kirsch *An introduction to the Mathematical Theory of Inverse Problem* Second Edition
- [6] A.A.Kilbas, H.M. Srivastava, J.J.Trujillo, Theory and Application of Fractional differnetial equations, North - Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.
- [7] A.A. Kilbas , H.M. Srivastava and J.J. Trujillo *Theory and Applications of Fractional Differential Equations*, (Amsterdam: Elsevier),2006.
- [8] M. Kirane, A.S. Malik, *Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time*, Appl. Math. Comput. 218 (2011), no. 1, 163-170.
- [9] J. J. Liu and M. Yamamoto, *A backward problem for the time-fractional diffusion equation*, Appl. Anal. 89 (2010), 1769-1788.
- [10] M.M. Meerschaert, E. Nane and P. Vellaisamy. Fractional Cauchy problems on bounded domains. *Ann. Probab.* 37 (2009), 979-1007.
- [11] D.A. Murio *Stable numerical solution of fractional-diffusion inverse heat conduction problem* Comput. Math. Appl. 53, 2007, 1492-501.
- [12] R.R. Nigmatullin. The realization of the generalized transfer in a medium with fractal geometry. *Phys. Status Solidi B.* 133 (1986) 425 – 430.
- [13] I.Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, vol 198, Academic Press Inc, San Diego, CA, 1990.
- [14] H.Pollard, The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$, Bull, Amer. Math.Soc. 54(12) (1948) 1115 - 1116.
- [15] K. Sakamoto, M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*. J. Math. Anal. Appl. 382 (2011), no. 1, 426-447.

- [16] N.H. Tuan, L.D. Long, N.V. Thinh, *Regularized solution of an inverse source problem for a time fractional diffusion equation*. Applied Mathematical Modelling, to appear.
- [17] T. Wei, J. Wang, *A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation* Appl. Numer. Math. 78 (2014), 95-111.
- [18] J.G. Wang, Y.B. Zhou, T. Wei, *Two regularization methods to identify a space-dependent source for the time-fractional diffusion equation* Appl. Numer. Math. 68 (2013), 3957.
- [19] Z.Q. Zhang, T. Wei, *Identifying an unknown source in time-fractional diffusion equation by a truncation method*. Appl. Math. Comput. 219 (2013), no. 11, 5972-5983.
- [20] G. Zaslavsky. Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D* **76** (1994) 110-122.